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Note

Use of The Simplex Method in Nonlinear Programming for Duct Layout Design*

Introduction

In recent years the complexity of fluid distribution systems, such as spacecraft environmental control systems and large-scale central air-conditioning systems, has increased to the point that design by engineering intuition may lead to costly results. Some researchers have considered the optimization of the components in a preselected configuration [1, 2], but little work has been done on the optimization of the configuration itself [3]. This paper discusses research [4] which has resulted in a new method which can be used to determine the optimal arrangement of components within a system. This optimal arrangement can be used as the basis for the final detailed design, including pressure balancing.

Specifically, the method presented optimizes the ducting arrangement of any fluid distribution system for which the objective function, the cost, is separable and concave with respect to flow rate and the required flow rate is known at given locations. The method involves a modified Simplex linear-programming procedure, which accommodates the nonlinear cost function by linearly approximating it in such a way that all possibly favorable (cost reducing) changes in the path arrangement are examined.

Description of Problem

The distribution network considered here consists of a single source and n sinks of arbitrary but specified capacity r_i, where i is the sink number. There are two connections between every pair of nodes (source and sinks), one in each direction, such that there are $m = 2\binom{n+1}{2} = n(n+1)$ connections or paths. Each path has a nonnegative flow rate x_j, where j is the path number.

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A flow balance at each node yields the constraint equations, which may be written as

$$\sum_{j=1}^{m} a_{ij}x_j = r_i \qquad i = 1, 2,..., n,$$

and (1)

$$x_j \geqslant 0 \qquad j = 1, 2,..., m,$$

where

$$a_{ij} = \begin{cases} +1 & \text{if the flow in the } j\text{th path is to the } i\text{th node,} \\ -1 & \text{if the flow in the } j\text{th path is away from the } i\text{th node,} \\ 0 & \text{if the } j\text{th path is not connected to the } i\text{th node.} \end{cases}$$

The objective function is

$$C = \sum_{j=1}^{m} c_j(x_j), \tag{2}$$

where $c_j(x_j)$ is the nonlinear but concave cost function for each individual path.

Analysis of Algorithm

Classical linear programming techniques [5, 6] cannot be used to solve the subject problem directly due to the nonlinear objective function. Nonlinear methods, such as polygonal approximation, may be used, but not without the expense of large increases in the number of variables and amount of computation [7].

If the objective function were linear, say

$$C = \sum_{i=1}^{m} \alpha_j x_j , \tag{3}$$

then at any step in the Simplex method we know that x_{k_1} enters the basis and x_{p_1} departs (provided we are not already at the optimum), where k_1 is defined by

$$y_{k_1} = t_{k_1} - \alpha_{k_1} = \max_i (t_i - \alpha_i) > 0, \tag{4}$$

$$t_i = \sum_{j=1}^{n} \alpha_j a_{ji} , \tag{5}$$

and p_1 is defined by

$$\theta = \frac{r_{p_1}}{a_{p_1 k_1}} = \min_j \frac{r_j}{a_{jk_1}} \geqslant 0. \tag{6}$$

(The applicability of the Simplex basic feasible solution for concave cost functions is indicated in Appendix A.) The new basic feasible solution is

$$\begin{aligned} x'_{k_1} &= \theta, \\ x_j' &= 0 \qquad \text{nonbasic } j, \text{ except } \ j = k_1, \\ x_j' &= x_j - \theta a_{jk_1} \qquad \text{basic } j. \end{aligned} \tag{7}$$

In the problem of interest, however, the cost function is not linear; so y_i poses a severe problem.

The new method described in the remainder of this paper involves linearly approximating the cost function at each step in the iteration and for each flow in the basis.

Physically, the selection of the subscript k_1 implies that the new duct will be one whose addition will produce the maximum rate of decrease in system cost. This selection can be accomplished in the nonlinear case by estimating the effects the addition of the duct will have on the costs of the other ducts, using

(1) the maximum possible rate of decrease in cost when the trial value of k_1 implies that the flow rate is reduced in a duct, and

(2) the minimum possible rate of increase in cost when the trial value of k_1 implies the flow rate is increased in a duct.

The combination of decreased and increased costs associated with all possible selections of k_1 leads, by direct comparison, to the recognition of the one combination of k_1 and p_1 [via Eqs. (4) and (6)] which maximizes the rate of decrease in system cost (provided the optimal solution has not already been found). The problem is to determine whether a particular choice of k_1 causes the flow to increase or decrease in each of the other ducts.

From the third part of Eq. (7), we can see that, if

$$a_{ik_1} = -1 < 0, \qquad \text{then } \ x_i' > x_i \text{ (increases)}, \tag{8}$$

$$a_{ik_1} = 0, \qquad \text{then } \ x_i' = x_i \text{ (no change)}, \tag{9}$$

$$a_{ik_1} = +1 > 0, \qquad \text{then } \ x_i' < x_i \text{ (decreases)}. \tag{10}$$

Simply by observing the sign of a_{ik_1} we can determine if the flow in a duct will increase or decrease by the addition of duct "k_1", and can therefore select the linear approximation to the cost function such that the nonlinear version of y_{k_1} overestimates the improvement in this cost, C.

It is necessary to collectively maximize the savings and minimize the additional costs due to the addition of x_q. Before x_q is introduced the value of x_i is w_0 with cost $C_0 = c_i(w_0)$.

If $a_{iq} = +1$, x_i will decrease when x_q is added and the savings is

$$\Delta C^- = C_0 - C_1 = c_i(w_0) - c_i(w_1). \tag{11}$$

Since C_1 is not known, a linear approximation (Fig. 1) is used,

$$\Delta C^-_{\text{approx}} = C_0 - \left[\frac{C_0}{w_0}\right] w_1 . \tag{12}$$

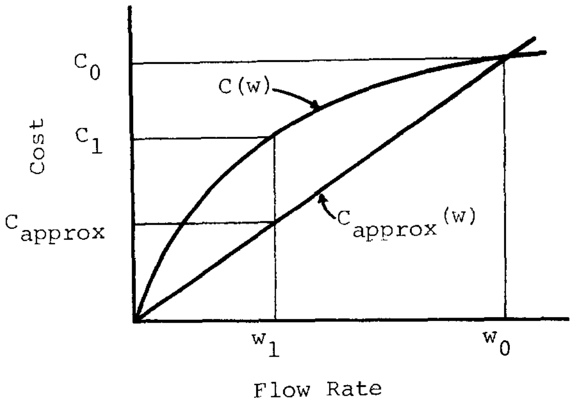

FIG. 1. Exact and approximate cost functions for $a_{iq} = +1$.

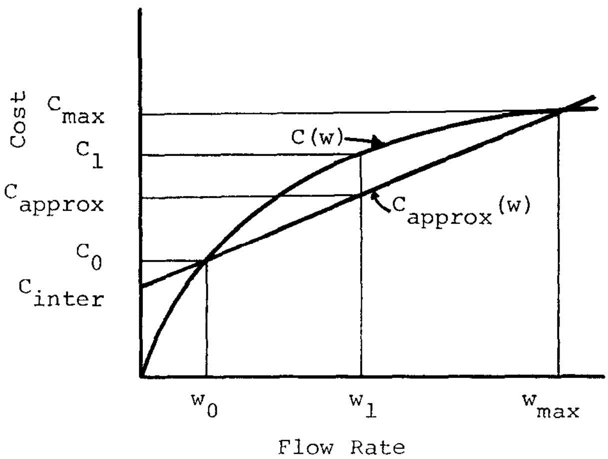

FIG. 2. Exact and approximate cost functions for $a_{iq} = -1$.

Since c_i is concave,

$$\Delta C^{-}_{\text{approx}} > \Delta C^{-}, \tag{13}$$

no matter what the magnitude of the change in x_i.

For $a_{iq} = -1$, x_i increases with the addition of x_q. The increase in cost for x_i is

$$\Delta C^{+} = C_1 - C_0 = c_i(w_1) - c_i(w_0). \tag{14}$$

The linear approximation (Fig. 2) is

$$\Delta C^{+}_{\text{approx}} = \left[\frac{C_{\max} - C_0}{w_{\max} - w_0}\right] w_1 + C_{\text{inter}} - C_0 , \tag{15}$$

where $w_{\max}$ is the total system flow. No matter what the value of x_i,

$$\Delta C^{+}_{\text{approx}} < \Delta C^{+}. \tag{16}$$

The above approximations provide slope estimates that can be used in the computation of t_i. The α_{i_0} term in $(t_i - \alpha_{i_0})$ can be estimated by the local slope at $w_{\max}$ so that the cost of adding the new duct is underestimated.

This procedure insures that all favorable changes will be attempted; however, unfavorable changes may appear to be favorable. For this reason, before a change is actually made, the total system cost under that change is compared with the present cost. If the new cost is less, the change is made. If the new cost is greater, the next most favorable change is made.

As in the Simplex method, the constraint equations must be rearranged so the a_{ij}'s corresponding to the constraint equations, $i = 1, 2,..., n$, and the ducts with nonzero flow, $j = 1, 2,..., p_1 - 1, p_1 + 1,..., n, k_1$, form the identity matrix. Since $a_{ij} = 0$ or ± 1 in the present case, the necessary transformations take the simple form

$$r_i' = r_i - \theta a_{ik_1}, \qquad i \neq p_1, \tag{17}$$

$$r'_{p_1} = \theta, \tag{18}$$

$$a'_{ij} = a_{ij} - a_{p_1 j} a_{ik_1}, \qquad i \neq p_1, \tag{19}$$

$$a'_{p_1 j} = a_{p_1 j}. \tag{20}$$

The algorithm can now be continued until the optimum is found.

Although convergence to an optimum has been established, convergence to the global optimum has not been proven. The nonlinear cost function prevents the adaptation of the proof used in linear programming to the current problem. The many examples that have been studied to date have all converged to the global optimum, but, of course, this is no proof. The procedure of overestimating the improvement in cost of all potential one-step changes gives credence to the idea that convergence to the global optimum will indeed occur for all cases.

CONCLUSION

A method has been developed which optimizes the path arrangement of a fluid distribution network when the demand is known at given locations and the cost function is separable and concave. It has been successfully implemented for use with air-conditioning–duct layout design. In addition, the method may have important applications in circuit design, routing problems, and transportation problems, in which the constraints are linear and the cost function is separable and concave.

APPENDIX A

The algorithm depends on the fact that it is always more economical to feed a node with a single duct rather than with more than one. This can be shown by assuming there is a node with total requirement w_0 which is more economically fed by $N + 1$ ducts rather than any single duct. Let

$$C = c_0\left(w_0 - \sum_{j=1}^{N} w_j\right) + \sum_{i=1}^{N} c_i(w_i). \tag{A-1}$$

Suppose $c_0(w_0)$ is the cost of the particular duct for which

$$c_0(w_0) < c_i(w_0), \qquad i = 1, 2,..., N. \tag{A-2}$$

Due to the concavity of the cost function,

$$c_i(\lambda w_0) \geqslant \lambda c_i(w_0), \qquad i = 0, 1,..., N. \tag{A-3}$$

Since we have assumed the $N + 1$ ducts are more economical than the single duct,

$$C < c_0(w_0). \tag{A-4}$$

But from Eqs. (A-1), (A-3), and (A-4),

$$c_0(w_0) > C > \frac{w_0 - \sum_{j=1}^{N} w_j}{w_0}\, c_0(w_0) + \sum_{i=1}^{N} \frac{w_i}{w_0}\, c_i(w_0). \tag{A-5}$$

Then

$$0 > \sum_{i=1}^{N} \frac{w_i}{w_0} [c_i(w_0) - c_0(w_0)], \tag{A-6}$$

which, since $w_i > 0$, contradicts Eq. (A-2).

Since $N = 1, 2, 3, \ldots$, it can be concluded that each node will be fed by one and only one duct. Furthermore, if only one duct most economically feeds one node, then n ducts most economically feed n nodes.

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